

# Existence and uniqueness of solutions of a class of 3<sup>rd</sup> order dissipative problems with various boundary conditions describing the Josephson effect

Monica De Angelis<sup>1</sup>, Gaetano Fiore<sup>1,2</sup>

<sup>1</sup> Dip. di Matematica e Applicazioni, Università “Federico II”  
V. Claudio 21, 80125 Napoli, Italy

<sup>2</sup> I.N.F.N., Sez. di Napoli, Complesso MSA, V. Cintia, 80126 Napoli, Italy

## Abstract

We prove existence and uniqueness of solutions of a large class of initial-boundary-value problems characterized by a quasi-linear third order equation (the third order term being dissipative) on a finite space interval with Dirichlet, Neumann or pseudoperiodic boundary conditions. The class includes equations arising in superconductor theory, such as a well-known modified sine-Gordon equation describing the Josephson effect, and in the theory of viscoelastic materials.

## 1 Introduction

In this paper we study the class of third order problems

$$\begin{aligned} Lu &= f(x, t, U), & L &:= \partial_t^2 + a\partial_t - c^2\partial_x^2[\varepsilon\partial_t + 1], & x \in \mathring{D}, \quad t > 0 \\ u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x) \end{aligned} \quad (1)$$

[we have abbreviated  $U := (u, u_x, u_t)$ ] where the domain  $D$  and the boundary conditions are respectively given by

$$\begin{aligned} D = [0, \pi], & \quad u(0, t) = h_0(t), \quad u(\pi, t) = h_\pi(t), & \text{DBC}, \\ D = [0, \pi], & \quad u_x(0, t) = k_0(t), \quad u_x(\pi, t) = k_\pi(t), & \text{NBC}, \\ D = \mathbb{R}, & \quad u(x + 2\pi, t) = u(x, t) + 2\pi m, & \text{PBC}. \end{aligned} \quad (2)$$

Here and in the sequel: we use DBC, NBC and PBC as abbreviations for Dirichlet, Neumann and (pseudo)periodic (with  $m \in \mathbb{Z}$ ) boundary conditions respectively;  $a, \varepsilon$  are respectively a real and a positive constant;  $f$  is continuous and in the PBC case fulfills the compatibility condition

$$f(x+2\pi, t, u+2\pi m, u_x, u_t) = f(x, t, u, u_x, u_t); \quad (3)$$

having set  $I := [0, \infty[$ ,  $u_0, u_1 \in C^2(D)$  [and fulfill  $(2)_3$  in the PBC case],  $h_0, h_\pi \in C^2(I)$ , [resp.  $k_0, k_\pi \in C^1(I)$ ] are assigned so as to be of period  $2\pi$  in the PBC case, otherwise so as to fulfill the consistency matching conditions

$$\begin{aligned} h_0(0) &= u_0(0), & \dot{h}_0(0) &= u_1(0), & h_\pi(0) &= u_0(\pi), & \dot{h}_\pi(0) &= u_1(\pi) & \text{DBC}, \\ k_0(0) &= u'_0(0), & \dot{k}_0(0) &= u'_1(0), & k_\pi(0) &= u'_0(\pi), & \dot{k}_\pi(0) &= u'_1(\pi) & \text{NBC}. \end{aligned} \quad (4)$$

The  $\varepsilon$ -term is dissipative; if  $a > 0$  the  $a$ -term is dissipative as well.

Theorems of existence and uniqueness of solutions of various versions of problem (1) on  $D = \mathbb{R}$  with  $u$  going to zero as  $x \rightarrow \pm\infty$  were given in [1, 2, 3, 4]. Theorems of existence and uniqueness for some version of problem (1) with DBC  $(2)_1$ , as well as for the qualitative properties (boundedness, stability, attractivity, ...) of the solutions, have been given in [5, 6, 7, 8].

In this work we prove general results concerning the existence and uniqueness for all positive  $t$  of the solution of (1) with PBC, DBC or NBC. In the proof we show also rather stringent properties of the fundamental solutions of the equation  $Lu = 0$  resp. fulfilling the PBC, DBC, NBC.



Figure 1: Josephson Junction (left) and schematic representation of a Voigt material (right)

Physically remarkable examples of problems (1-2) include:

- If  $f = b \sin u - \gamma$ , with  $b, \gamma = \text{const}$ , a modified sine-Gordon eq. describing **Josephson effect** [9] in the theory of superconductors, which is at the base (see e.g. [10]) of a large

number of advanced developments both in fundamental research (e.g. macroscopic effects of quantum physics, quantum computation) and in applications to electronic devices (see e.g. Chapters 3-6 in [11]):  $u(x, t)$  is the phase difference of the macroscopic wavefunctions of the Bose-Einstein condensate of Cooper pairs in two superconductors separated by a *Josephson junction* (JJ), i.e. a very thin and narrow dielectric strip of finite length (Fig. 1-left), the term  $au_t$  is due to the Joule effect of the residual current of single electrons across the JJ, the term  $\varepsilon u_{xxt}$  is due to the surface impedance of the JJ. In the simplest model adopted to describe the JJ  $a = 0$  and  $\varepsilon, c^2 = \text{const}$  ( $\varepsilon$  is rather small); more accurately,  $a$  is positive but very small; even more accurately, one adopts  $f = b \sin u - \gamma + a(1 - \cos u)u_t$ . The equation must be complemented by NBC if the strip is open, by PBC if the strip is closed in the form of a ring. In the latter case the topological invariant  $m$  counts the number of *fluxons* (i.e. quanta of magnetic flux) which may move along the ring but remain trapped in it, as the closed lines of the magnetic field passing through the junction and encircling one of the two superconductors cannot escape crossing it, by the Meissner effect.

- If  $a=0$ ,  $f = f(x, t)$ , an equation (see e.g. [12, 1]) for the displacement  $u(x, t)$  of the section of a rod from its rest position  $x$  in a Voigt material:  $f$  is applied density force,  $c^2 = E/\rho$ ,  $\varepsilon = 1/\rho\mu$ , where  $\rho$  is the linear density of the rod at rest,  $E, \mu$  are the elastic and viscous constants of the rod, which enter the stress-strain relation  $\sigma = E\nu + \partial_t \nu/\mu$ , where  $\sigma$  is the stress  $\nu$  is the strain (as known, a discretized model of the rod is a series of elements consisting of a viscous damper and an elastic spring connected in parallel as shown in Fig. 1-right).
- Equations used to describe: heat conduction at low temperature  $u$  [13, 14], if  $\varepsilon = c^2$ ,  $f = 0$ ; sound propagation in viscous gases [15]; propagation of plane waves in perfect incompressible and electrically conducting fluids [16].

We may assume without loss of generality that  $c^2 = 1$  [this can be always obtained by the rescaling (10) with  $a = 0$  of  $x$ ]. Redefining

$$\begin{aligned} \hat{u}(x, t) &= u(x, t) - \phi_m(x), & \text{PBC,} \\ \hat{u}(x, t) &= u(x, t) + \left[ \frac{x^2}{2\pi} - x \right] k_0(t) - \frac{x^2}{2\pi} k_\pi(t), & \text{NBC,} \\ \hat{u}(x, t) &= u(x, t) + \left[ \frac{x}{\pi} - 1 \right] h_0(t) - \frac{x}{\pi} h_\pi(t), & \text{DBC,} \end{aligned} \tag{5}$$

where  $\phi_m(x)$  is such that  $\phi_m(x+2\pi) = \phi_m(x) + 2\pi m$ , e.g.  $\phi_m(x) := mx$ , we find that  $\hat{u}$  fulfills the PDE, initial conditions

$$L\hat{u} = \hat{f}, \quad \hat{u}(x, 0) = \hat{u}_0(x), \quad \hat{u}_t(x, 0) = \hat{u}_1(x) \tag{6}$$

and the boundary conditions

$$\begin{aligned} \hat{u}(x+2\pi, t) &= \hat{u}(x, t), & \text{PBC,} \\ \hat{u}(0, t) &= 0, & \hat{u}(\pi, t) = 0, & \text{DBC,} \\ \hat{u}_x(0, t) &= 0, & \hat{u}_x(\pi, t) = 0, & \text{NBC,} \end{aligned} \quad (7)$$

with  $\hat{f}$  and initial conditions respectively given by

$$\begin{aligned} \hat{f}(x, t, \hat{u}, \hat{u}_x, \hat{u}_t) &= f[x, t, \hat{u} + \phi^m(x), \hat{u}_x, \hat{u}_t] + \phi_{xx}^m(x), & \hat{u}_0 &= u_0 - \phi^m(x), & \hat{u}_1 &= u_1 & \text{PBC,} \\ \hat{f} &= f + \left[\frac{x}{\pi} - 1\right](\ddot{h}_0 + a\dot{h}_0) - \frac{x}{\pi}(\ddot{h}_\pi + a\dot{h}_\pi), & \begin{cases} \hat{u}_0 = u_0 + \left[\frac{x}{\pi} - 1\right]h_0(0) - \frac{x}{\pi}h_\pi(0) \\ \hat{u}_1 = u_1 + \left[\frac{x}{\pi} - 1\right]h_0(0) - \frac{x}{\pi}h_\pi(0) \end{cases} & \text{DBC,} \\ \hat{f} &= f + \left[\frac{x^2}{2\pi} - x\right](\ddot{k}_0 + a\dot{k}_0) - \frac{x^2}{2\pi}(\ddot{k}_\pi + a\dot{k}_\pi), & \begin{cases} \hat{u}_0 = u_0 + \left[\frac{x^2}{2\pi} - x\right]k_0(0) - \frac{x^2}{2\pi}k_\pi(0) \\ \hat{u}_1 = u_1 + \left[\frac{x^2}{2\pi} - x\right]k_0(0) - \frac{x^2}{2\pi}k_\pi(0) \end{cases} & \text{NBC.} \end{aligned} \quad (8)$$

$\hat{u}_0, \hat{u}_1$  automatically fulfill the consistency conditions

$$\begin{aligned} \hat{u}_0(0) &= 0, & \hat{u}_1(0) &= 0, & \hat{u}_0(\pi) &= 0, & \hat{u}_1(\pi) &= 0 & \text{DBC,} \\ \hat{u}'_0(0) &= 0, & \hat{u}'_1(0) &= 0, & \hat{u}'_0(\pi) &= 0, & \hat{u}'_1(\pi) &= 0 & \text{NBC.} \end{aligned} \quad (9)$$

Consequently, without loss of generality we can assume in (2)  $m=0$ ,  $h_i \equiv 0$  and  $k_i \equiv 0$  ( $i=0, \pi$ ) respectively for the PBC, DBC, NBC, namely assume the boundary conditions (7), (9). Note that in the PBC case from (3) it follows  $\hat{f}(x+2\pi, t, U) = \hat{f}(x, t, U)$ , as it must be. We shall remove the superscripts  $\hat{\phantom{x}}$  henceforth.

We may also assume without loss of generality that  $a \geq 0$ ,  $c = 1$ . In fact, if  $a < 0$  then  $1 - \frac{a}{2}\varepsilon > 0$ , so that redefining

$$\begin{aligned} \tilde{c} &:= c\sqrt{1 - \frac{a}{2}\varepsilon}, & \tilde{x} &:= \frac{1}{\tilde{c}}x, & \tilde{\varepsilon} &:= \frac{\varepsilon}{1 - \frac{a}{2}\varepsilon}, & \tilde{L} &:= \partial_t^2 - \partial_{\tilde{x}}^2(\tilde{\varepsilon}\partial_t + 1), \\ \tilde{u}(\tilde{x}, t) &:= e^{\frac{a}{2}t}u(\tilde{c}\tilde{x}, t), & \tilde{u}_0(x) &:= u_0(\tilde{c}\tilde{x}), & \tilde{u}_1(x) &:= u_1(\tilde{c}\tilde{x}), \\ \tilde{f}(\tilde{x}, t, \tilde{u}, \tilde{u}_{\tilde{x}}, \tilde{u}_t) &:= \frac{a^2}{4}\tilde{u} + e^{\frac{a}{2}t}f\left[\tilde{c}\tilde{x}, t, e^{\frac{-a}{2}t}\tilde{u}, \frac{e^{-\frac{a}{2}t}}{\tilde{c}}\tilde{u}_{\tilde{x}}, e^{\frac{-a}{2}t}(\tilde{u}_t - \frac{a}{2}\tilde{u})\right], \end{aligned} \quad (10)$$

we find that  $\tilde{u}$  fulfills the PDE, initial conditions

$$\tilde{L}\tilde{u} = \tilde{f}, \quad \tilde{u}(\tilde{x}, 0) = \tilde{u}_0(\tilde{x}), \quad \tilde{u}_t(\tilde{x}, 0) = \tilde{u}_1(\tilde{x}), \quad (11)$$

and again boundary conditions of the type (7), (9). We shall remove the superscripts  $\tilde{\phantom{x}}$  henceforth.

## 2 The fundamental solutions of $Lu = 0$

By saying that  $v(x, t)$  is a solution of  $Lv = 0$  we mean that  $v, v_t, v_{tt}, \partial_x^2(\varepsilon v_t + v)$  are continuous and the combination  $Lv$  is zero for  $t > 0$ . Any solution  $u^d$  of  $Lu = 0$  and the DBC (7) [resp.  $u^n$  of  $Lu = 0$  and the NBC (7)] can be transformed by an odd (resp. even) extension into a solution  $u^p$  of  $Lu = 0$  and the PBC (7), as follows. As a first step,

$$\begin{aligned} u^p(x, t) &:= \begin{cases} u^d(x, t) & x \in [0, \pi] \\ -u^d(-x, t) & x \in ]-\pi, 0[ \end{cases} & \text{DBC,} \\ u^p(x, t) &:= \begin{cases} u^n(x, t) & x \in [0, \pi] \\ u^n(-x, t) & x \in ]-\pi, 0[ \end{cases} & \text{NBC;} \end{aligned} \quad (12)$$

as a second step, in either case setting for  $x \in ]-\pi, \pi]$  and any  $k \in \mathbb{Z}$

$$u^p(x + 2k\pi, t) := u^p(x, t). \quad (13)$$

It is immediate to check that, because of (7),  $u^p, u_x^p$  and their first and second time derivatives are continuous at all points  $x = k\pi$ ; moreover, because of  $Lu^d = 0$  (resp.  $Lu^n = 0$ ), then  $Lu^p = 0$  everywhere and, since  $u_t^p, u_{tt}^p$  are continuous also at all points  $x = k\pi$ , also  $\partial_x^2(\varepsilon u_t^p + u^p)$  is continuous, as claimed. Therefore the fundamental solutions of  $Lu = 0$  and the DBC, NBC (7) can be extended as particular solutions of  $Lu = 0$  and the PBC (7).

For all  $n \in \mathbb{Z}$  the products  $v_n(x, t) = H_n(t)e^{inx}$  are periodic solutions of  $Lv = 0$  provided

$$\ddot{H}_n + a\dot{H}_n + n^2(\varepsilon\dot{H}_n + H_n) = 0; \quad (14)$$

then also  $v_{nt}$  are. We choose the solutions fulfilling the initial conditions  $H_n(0) = 0, \dot{H}_n(0) = 1$ :

$$H_n(t) := e^{-h_n t} \frac{\sinh(\omega_n t)}{\omega_n}, \quad h_n = \frac{a + \varepsilon n^2}{2}, \quad \omega_n = \sqrt{h_n^2 - n^2}, \quad (15)$$

in particular  $H_0(t) := \frac{1 - e^{-at}}{a}$ ;  $H_n$  must be understood as its  $\omega_n \rightarrow 0$  limit when  $\omega_n = 0$ :

$$H_n(t) := e^{-h_n t} t \quad \text{if } \omega_n = 0. \quad (16)$$

Clearly  $h_{-n} = h_n, \omega_{-n} = \omega_n, H_{-n} = H_n$ ; note that  $H_n$  is real even when  $\omega_n$  is imaginary. Any finite combination of the  $v_n, v_{nt}$  is a periodic solution of  $Lv = 0$ . We now inquire if also the following (Fourier) series defines one:

$$\vartheta(x, t) := \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} H_n(t) e^{inx} = \frac{1}{2\pi} H_0(t) + \frac{1}{\pi} \sum_{n=1}^{\infty} H_n(t) \cos(nx). \quad (17)$$

**Proposition 2.1** *If  $a \geq 0$  the series (17) defines for all  $t \geq 0$  a continuous, real-valued function  $\vartheta(x, t)$ , even and of period  $2\pi$  w.r.t.  $x$ , such that  $\vartheta(x, 0) \equiv 0$ , bounded as follows:*

$$2\pi|\vartheta(x, t)| \leq 2\pi\vartheta(0, t) \leq N(t) := M + \begin{cases} a^{-1} & \text{if } a > 0, \\ t & \text{if } a = 0, \end{cases} \quad (18)$$

$$M := 2 + 2 \log \bar{n} + \frac{2\pi^2}{3\varepsilon}, \quad \bar{n} := 1 + \left\lfloor \frac{2}{\varepsilon} \right\rfloor; \quad (19)$$

here  $[y]$  means the integer part of  $y$ . For  $t > 0$  the derivatives  $\vartheta_t, \vartheta_{tt}, \vartheta_{ttt}, \partial_x^2(\varepsilon\vartheta_t + \vartheta), \partial_x^2(\varepsilon\vartheta_{tt} + \vartheta_t)$  are well-defined, equal the term-by-term derived series and fulfill

$$L\vartheta = 0, \quad L\vartheta_t = 0 \quad (20)$$

for all  $(x, t) \in \mathbb{R} \times \mathbb{R}^+$ . Finally,  $\vartheta_x(\cdot, t) \in L^2([0, \pi])$  for all  $t \geq 0$ ,  $\vartheta_t(\cdot, t), \vartheta_{xt}(\cdot, t) \in L^2([0, \pi])$  for all  $t > 0$ , with square  $L^2$ -norms bounded as follows:

$$\begin{aligned} 4\pi^2 \|\vartheta_x(\cdot, t)\|_2^2 &< 2 + \frac{4}{\varepsilon} + \frac{4\pi^2}{3\varepsilon^2}, \\ 4\pi^2 \|\vartheta_t(\cdot, t)\|_2^2 &< \kappa + 8e^{\frac{4t}{\varepsilon}} \theta(0, i\frac{2}{\pi}\varepsilon t), \quad \kappa := 3 + \frac{4}{\varepsilon} + \frac{2\pi^2}{9\varepsilon^2}, \\ 4\pi^2 \|\vartheta_{tx}(\cdot, t)\|_2^2 &< \left(\frac{2}{\varepsilon} + 1\right)^4 \left[\left(\frac{2}{\varepsilon} + 1\right)^4 + 1\right] + \frac{12}{\varepsilon^2} - \frac{8}{\varepsilon} e^{\frac{4t}{\varepsilon}} \partial_t [\theta(0, i\frac{2}{\pi}\varepsilon t)]; \end{aligned} \quad (21)$$

here  $\|g\|_2^2 := \int_0^{2\pi} \frac{dx}{2\pi} |g(x)|^2$ , and  $\theta(z, \tau) := \sum_{n \in \mathbb{Z}} e^{i\pi(2nz + n^2\tau)}$  is Jacobi Theta function.

We recall that for  $\eta > 0$   $\theta(0, i\eta) > 0$  and  $\theta(0, i\eta) \sim \eta^{-\frac{1}{2}}$  as  $\eta \downarrow 0$ .

If  $a < 0$  one could show that the above bounds hold adding at the rhs some term proportional to  $e^{-at}$  (which is increasing with  $t$ ), while all other claims hold unmodified. We don't need to do, because by the change of variables (10) we reduce the existence and uniqueness theorem for the case  $a < 0$  to the one for the case  $a = 0$  (see section 4).

In order to prove the proposition we first prove

**Lemma 2.1** *If  $a \geq 0$   $H_n$  and its time derivatives fulfill the following bounds:*

$$\begin{aligned} \left| \frac{d^l H_n}{dt^l} \right| &\leq \frac{1}{2\omega_n} \left[ \frac{2^l}{\varepsilon^l} + (a + \varepsilon n^2)^l e^{-t(\varepsilon n^2 - \frac{2}{\varepsilon})} \right] \\ n^2 |\varepsilon \dot{H}_n + H_n| &\leq \frac{1}{2\omega_n} \left[ \frac{a\varepsilon + 4}{\varepsilon^2 n^2} + n^2 (a\varepsilon + \varepsilon^2 n^2 + 1) e^{-t(\varepsilon n^2 - \frac{2}{\varepsilon})} \right] \quad \text{if } |n| \geq \bar{n}, \\ n^2 |\varepsilon \ddot{H}_n + \dot{H}_n| &\leq \frac{1}{2\omega_n} \left[ \frac{8 + 2a\varepsilon}{\varepsilon^3} + n^2 (a\varepsilon + \varepsilon^2 n^2 + 1)(a + \varepsilon n^2) e^{-t(\varepsilon n^2 - \frac{2}{\varepsilon})} \right] \end{aligned} \quad (22)$$

$$|H_n(t)| \leq \begin{cases} \frac{1}{|n|} & \text{if } 0 < |n| \leq \bar{n}, \\ \frac{2}{\varepsilon} \frac{1}{n^2} & \text{if } |n| \geq \bar{n}. \end{cases} \quad (23)$$

$$|H_n(t)| \leq t, \quad (24)$$

$$|\dot{H}_n(t)| \leq 1, \quad (25)$$

$$|1 - \dot{H}_n(t)| < (2h_n + |\Im(\omega_n)|) t. \quad (26)$$

Clearly the bounds (24), (26) are stringent for  $t \sim 0$ .

*Proof.* If  $l = 0, 1, 2, \dots$  and  $\omega_n \neq 0$  we find

$$\begin{aligned} \frac{d^l H_n}{dt^l} &= \frac{1}{2\omega_n} [(\omega_n - h_n)^l e^{(\omega_n - h_n)t} - (-)^l (\omega_n + h_n)^l e^{-(\omega_n + h_n)t}], \\ \varepsilon \dot{H}_n + H_n &= \frac{1}{2\omega_n} [(1 + \varepsilon\omega_n - \varepsilon h_n) e^{(\omega_n - h_n)t} + (\varepsilon\omega_n + \varepsilon h_n - 1) e^{-(\omega_n + h_n)t}], \\ \varepsilon \ddot{H}_n + \dot{H}_n &= \frac{1}{2\omega_n} [(1 + \varepsilon\omega_n - \varepsilon h_n)(\omega_n - h_n) e^{(\omega_n - h_n)t} - (\varepsilon\omega_n + \varepsilon h_n - 1)(\omega_n + h_n) e^{-(\omega_n + h_n)t}], \end{aligned} \quad (27)$$

whence

$$\begin{aligned} \left| \frac{d^l H_n}{dt^l} \right| &\leq \frac{1}{2|\omega_n|} [|h_n - \omega_n|^l |e^{(\omega_n - h_n)t}| + |\omega_n + h_n|^l |e^{-(\omega_n + h_n)t}|], \\ n^2 |\varepsilon \dot{H}_n + H_n| &\leq \frac{n^2}{2|\omega_n|} [|1 + \varepsilon\omega_n - \varepsilon h_n| |e^{(\omega_n - h_n)t}| + |\varepsilon\omega_n + \varepsilon h_n - 1| |e^{-(\omega_n + h_n)t}|], \\ n^2 |\varepsilon \ddot{H}_n + \dot{H}_n| &\leq \frac{n^2}{2|\omega_n|} [|1 + \varepsilon\omega_n - \varepsilon h_n| |h_n - \omega_n| |e^{(\omega_n - h_n)t}| \\ &\quad + |\varepsilon\omega_n + \varepsilon h_n - 1| |\omega_n + h_n| |e^{-(\omega_n + h_n)t}|]. \end{aligned} \quad (28)$$

We recall that for  $0 < \sigma < 1$

$$\left. 1 - \frac{\sigma}{2} - \frac{\sigma^2}{2} \right\} < \sqrt{1 - \sigma} < 1 - \frac{\sigma}{2}; \quad (29)$$

these inequalities follow from their squares, and become equalities for  $\sigma = 0$ . Clearly,  $h_n \rightarrow \infty$ ,  $\frac{n^2}{h_n^2} \rightarrow 0$  as  $|n| \rightarrow \infty$ , hence there exists a  $\bar{n} \in \mathbb{N}$  such that

$$|n| \geq \bar{n} \quad \Rightarrow \quad h_n > 0, \quad \frac{n^2}{h_n} \leq \frac{2}{\varepsilon}, \quad \sigma_n := \frac{n^2}{h_n^2} \in ]0, 1[, \quad \frac{\omega_n}{h_n} = \sqrt{1 - \sigma_n} > 0. \quad (30)$$

Note also that for  $n \geq \bar{n}$  the sequences  $h_n, \frac{\omega_n}{h_n}$  are increasing with  $n$ , while the sequence  $\sigma_n$  is decreasing. One can choose  $\bar{n}$  as in (19).<sup>1</sup> From (29-30) we find for  $|n| \geq \bar{n}$

$$\left. 1 - \frac{n^2}{2h_n^2} - \frac{n^4}{2h_n^4} \right\} \leq \frac{\omega_n}{h_n} \leq 1 - \frac{n^2}{2h_n^2}, \quad (32)$$

$$0 \leq \frac{n^2}{2h_n} \leq h_n - \omega_n \leq \begin{cases} \frac{n^2}{h_n} \leq \frac{2}{\varepsilon} \\ \frac{n^2}{2h_n} + \frac{n^4}{2h_n^3}, \end{cases} \quad (33)$$

$$\left. a + \varepsilon n^2 - \frac{2}{\varepsilon} \right\} \leq \omega_n + h_n \leq a + \varepsilon n^2, \quad (34)$$

$$\begin{aligned} \frac{a - \frac{4}{\varepsilon}}{a + \varepsilon n^2} &\leq \frac{a - \frac{n^4 \varepsilon}{h_n^2}}{a + \varepsilon n^2} = 1 - \frac{n^2 \varepsilon}{2h_n} - \frac{n^4 \varepsilon}{2h_n^3} \\ &\leq 1 + \varepsilon(\omega_n - h_n) \leq 1 - \frac{n^2 \varepsilon}{2h_n} = \frac{a}{a + \varepsilon n^2}, \\ \Rightarrow |1 + \varepsilon(\omega_n - h_n)| &\leq \frac{a\varepsilon + 4}{\varepsilon^2 n^2}, \end{aligned} \quad (35)$$

Each of the inequalities (33-35) is based on the preceding ones, (15) or  $\omega_n + h_n = \omega_n - h_n + 2h_n$ . Formulae (28-35) imply (22).

We now better evaluate the upper bound on  $H_n^2(t)$ . Except in the case  $a = n = 0$ , this is a smooth function vanishing at  $t = 0$  and going to zero as  $t \rightarrow \infty$ . Its maximum is reached at the smallest solution<sup>2</sup>  $t = t_n \geq 0$  of the eq.  $\dot{H}_n(t) = 0$ , i.e. of

$$\begin{aligned} e^{2\omega_n t_n} &= \frac{h_n + \omega_n}{h_n - \omega_n} = \frac{(h_n + \omega_n)^2}{n^2} && \text{if } \omega_n \neq 0, \\ t_n &= \frac{1}{h_n} && \text{if } \omega_n = 0 \end{aligned}$$

whence it follows in either case<sup>3</sup>

$$|H_n(t)| \leq |H_n(t_n)| = e^{-h_n t_n} |n|^{-1}. \quad (36)$$

<sup>1</sup> The first two inequalities (30) are automatic. The fourth relation is a consequence of the third. The latter holds iff  $\bar{n}/h_{\bar{n}} < 1$ , or equivalently  $\varepsilon \bar{n}^2 - 2\bar{n} + a > 0$ ; this is satisfied by all  $\bar{n} \geq 1$  if  $a\varepsilon > 1$ , because then the solutions

$$n_{\pm} = (1 \pm \sqrt{1 - a\varepsilon})/\varepsilon \quad (31)$$

of the equation  $\varepsilon m^2 - 2m + a = 0$  are not real; otherwise it is satisfied if we choose  $\bar{n} > n_+$ , in particular (19).

<sup>2</sup>In fact, there is only one solution for  $n$  such that  $\omega_n$  is real.

<sup>3</sup>In fact, if  $\omega_n \neq 0$  we find

$$|H_n(t_n)| = \frac{e^{-h_n t_n} |e^{-\omega_n t_n}|}{2|\omega_n|} |e^{2\omega_n t_n} - 1| = \frac{e^{-h_n t_n} |n|}{2|\omega_n(h_n + \omega_n)|} \left| \frac{(h_n + \omega_n)^2}{n^2} - 1 \right|$$



From (36) and  $h_n t_n \geq 0$  it follows (23) with  $|n| \leq \bar{n}$ . For  $|n| \geq \bar{n}$   $\omega_n$  is real and we find

$$e^{-h_n t_n} = [e^{-\omega_n t_n}]^{h_n/\omega_n} = \left[ \frac{|n|}{h_n + \omega_n} \right]^{h_n/\omega_n} \leq \frac{|n|}{h_n}, \quad (37)$$

whence (23) follows by (36). On the other hand, we recall the inequality<sup>4</sup>

$$|1 - e^{-z}| \leq |z| \quad \text{if } \Re z \geq 0 \quad (39)$$

(the inequality is strict iff  $z \neq 0$ ); applying it to relation (15) we find for all  $n \in \mathbb{Z}$   $|H_n(t)| \leq \left| \frac{1 - e^{-2\omega_n t}}{2\omega_n} \right| \leq t$ , i.e. (24).

We now better evaluate the bounds on  $\dot{H}_n(t)$ . From the definitions (15-16) it easily follows

$$\dot{H}_n(t) = e^{-h_n t} \cosh(\omega_n t) - h_n H_n(t) \quad (40)$$

For any  $n \in \mathbb{Z}$  this is a bounded function equal to 1 at  $t = 0$  and going to 0 as  $t \rightarrow \infty$ . Its infimum and supremum are reached either at 0,  $\infty$  or at the solutions  $t = t'_n \geq 0$  of the eq.  $\ddot{H}_n|_{t=t'_n} = 0$ , i.e. of

$$\begin{aligned} e^{2\omega_n t'_n} &= \left( \frac{h_n + \omega_n}{h_n - \omega_n} \right)^2 = \left[ \frac{(h_n + \omega_n)^2}{n^2} \right]^2 & \Leftrightarrow & e^{\omega_n t'_n} = \pm \left( \frac{h_n + \omega_n}{|n|} \right)^2 & \text{if } \omega_n \neq 0, \\ t'_n &= \frac{2}{h_n} & & & \text{if } \omega_n = 0 \end{aligned}$$

(in fact, there is only one solution if  $n$  is such that  $\omega_n$  is real; the minus sign in the first line may occur only if  $\omega_n$  is imaginary). Using (40) we find:  $\dot{H}_n(t'_n) = e^{-2} - 2e^{-2} = -e^{-2}$  if

---


$$= \frac{e^{-h_n t_n} |n|^{-1}}{2|\omega_n(h_n + \omega_n)|} |h_n^2 + \omega_n^2 + 2h_n \omega_n - n^2| = e^{-h_n t_n} |n|^{-1},$$

whereas if  $\omega_n = 0$  it is  $h_n = |n|$  and again  $|H_n(t_n)| = t_n e^{-h_n t_n} = h_n^{-1} e^{-h_n t_n} = e^{-h_n t_n} |n|^{-1}$ .

<sup>4</sup>Set  $z = x + iy$ . (39) can be proved in three steps:

$$x > 0 \quad \Rightarrow \quad e^{-x} < 1 \quad \Rightarrow \quad 1 - e^{-x} = \int_0^x dx' e^{-x'} < \int_0^x dx' = x \quad (38)$$

$$y > 0 \quad \Rightarrow \quad \sin y < y \quad \Rightarrow \quad 0 \leq 1 - \cos y = \int_0^y dy' \sin y' < \frac{y^2}{2}$$

$$|1 - e^{-z}|^2 = (1 - e^{-x-iy})(1 - e^{-x+iy}) = (1 - e^{-x})^2 + 2e^{-x}(1 - \cos y) < x^2 + y^2 = |z|^2.$$

$\omega_n = 0$ , and

$$\begin{aligned}\dot{H}_n(t'_n) &= \pm \frac{e^{-h_n t'_n}}{2\omega_n} \left[ (\omega_n + h_n) \frac{n^2}{(h_n + \omega_n)^2} + (\omega_n - h_n) \frac{(h_n + \omega_n)^2}{n^2} \right] \\ &= \pm \frac{e^{-h_n t'_n}}{2\omega_n} \frac{n^2 - (h_n + \omega_n)^2}{h_n + \omega_n} = \mp \frac{e^{-h_n t'_n}}{2\omega_n(h_n + \omega_n)} [h_n^2 + \omega_n^2 + 2h_n\omega_n - n^2] = \mp e^{-h_n t'_n}\end{aligned}$$

if  $\omega_n \neq 0$ . It follows in either case (25). Next, we show that

$$\begin{aligned}0 < 1 - \dot{H}_n(t) &< 2h_n t && \text{if } \omega_n \geq 0, \\ 0 < 1 - \dot{H}_n(t) &< 2h_n t + 2 \sin^2 \left( \frac{\Im(\omega_n)}{2} t \right) && \text{if } \omega_n = i\Im(\omega_n) \equiv i\sqrt{n^2 - h_n^2} \in i\mathbb{R}.\end{aligned}\tag{41}$$

The first/third inequality in (41) was already proved in (25). Using (40) and  $\cosh(\omega_n t) \geq 0$  we find  $1 - \dot{H}_n(t) \leq 1 - e^{-h_n t} + h_n H_n(t)$ ; the second inequality then follows by (39), (24). If  $\omega_n \in i\mathbb{R}$  then (40) gives

$$1 - \dot{H}_n(t) = 1 - e^{-h_n t} + h_n H_n(t) + e^{-h_n t} 2 \sin^2 \left( \frac{\sqrt{n^2 - h_n^2}}{2} t \right);$$

this yields the fourth inequality in (41) by (39), (24). Finally, (26) follows from (41) and  $|\sin y| \leq \min\{|y|, 1\}$ .  $\square$

*Proof of Proposition 2.1.* Using (22) it is easy to check that for any  $t > 0$  the Fourier series (17), all its term-by-term time derivatives  $\partial_t, \partial_t^2, \dots$ , as well as its term-by-term  $\partial_x^2(\varepsilon\partial_t + \text{id}), \partial_x^2(\varepsilon\partial_t^2 + \partial_t)$  derivatives, converge absolutely and uniformly in  $x$ ; consequently, (17) defines a continuous function  $\vartheta(x, t)$  whose derivatives  $\vartheta_t, \vartheta_{tt}, \dots, \partial_x^2(\varepsilon\vartheta_t + \vartheta), \partial_x^2(\varepsilon\vartheta_{tt} + \vartheta_t)$  are well-defined and equal to the sum of these series, with  $L\vartheta = L\vartheta_t = 0$ . In fact, from (17)

$$\begin{aligned}2\pi\partial_t^l\vartheta(x, t) &= \sum_{n=-m}^m \frac{d^l H_n}{dt^l} e^{inx} + R_m^l(x, t), \quad R_m^l(x, t) := \sum_{|n|>m} \frac{d^l H_n}{dt^l} e^{inx}, \\ |R_m^l(x, t)| &\leq \sum_{|n|>m} \left| \frac{d^l H_n}{dt^l} \right| \leq \sum_{|n|>\bar{n}} \frac{1}{2\omega_n} \left[ \frac{2^l}{\varepsilon^l} + (a + \varepsilon n^2)^l e^{-t(\varepsilon n^2 - \frac{2}{\varepsilon})} \right]\end{aligned}$$

( $m \in \mathbb{N}$ ); the inequalities in the second line and the fact that  $\frac{1}{\omega_n} \sim \frac{1}{h_n} \sim \frac{1}{n^2}$  as  $|n| \rightarrow \pm\infty$  show that, for all  $t > 0$  and  $l = 0, 1, \dots$ , the rest  $R_m^l(x, t)$  goes to zero as  $m \rightarrow \infty$  uniformly in  $x$ . This shows the absolute and uniform (in  $x$ ) convergence of the series (17) and all its term-by-term time derivatives. Similarly one proceeds for  $\partial_x^2(\varepsilon\vartheta_t + \vartheta), \partial_x^2(\varepsilon\vartheta_{tt} + \vartheta_t)$ .

Eq. (23) implies  $\sum_{n=1}^{\bar{n}} |H_n(t)| \leq 1 + \sum_{n=1}^{\bar{n}-1} \int_n^{n+1} dy \frac{1}{y} = 1 + \int_1^{\bar{n}} dy \frac{1}{y} = 1 + \log \bar{n}$  and

$$2\pi|\vartheta(x, t)| \leq 2\pi\vartheta(0, t) \leq \sum_{n=1-\bar{n}}^{\bar{n}-1} |H_n| + \sum_{|n| \geq \bar{n}} |H_n| \leq |H_0| + 2 + 2\log \bar{n} + \frac{2}{\varepsilon} \sum_{|n| \geq \bar{n}} \frac{1}{n^2};$$

the bound (18) follows by (15), (19) and  $\sum_{|n| \geq \bar{n}} \frac{1}{n^2} \leq 2\zeta(2) = \frac{\pi^2}{3}$ , a property of Riemann zeta

function  $\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}$ . Definition (19) implies  $\frac{2}{\varepsilon} + 1 \geq \bar{n} > \frac{2}{\varepsilon}$  and by (15)

$$\omega_n^2 \geq n^2 \left( \frac{\varepsilon^2 n^2}{4} - 1 \right) \geq \begin{cases} n^2 \left( \frac{\varepsilon^2 \bar{n}^2}{4} - 1 \right) > 3n^2, & \text{if } |n| \geq \bar{n} \\ n^2 \left( \frac{\varepsilon^2 \bar{n}^2 |n|^{3/2}}{4} - 1 \right) > n^2 (|n|^{\frac{3}{2}} - 1) > n^2 |n-1|^{\frac{3}{2}}, & \text{if } |n| \geq \bar{n}^4. \end{cases} \quad (42)$$

We prove (21) using (22-25), (36), (42),  $\zeta(2) = \frac{\pi^2}{6}$ ,  $\zeta(\frac{3}{2}) < 3$  and  $|n| \geq \bar{n} \Rightarrow \frac{h_n}{\omega_n} \leq 2$ :

$$\begin{aligned} 4\pi^2 \|\vartheta_x(\cdot, t)\|_2^2 &= \sum_{n \in \mathbb{Z}} |H_n(t)n|^2 \leq 2 \sum_{n=1}^{\bar{n}} 1 + 2 \sum_{n=\bar{n}+1}^{\infty} \frac{4}{\varepsilon^2 n^2} < 2\bar{n} + \frac{8}{\varepsilon^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < 2 + \frac{4}{\varepsilon} + \frac{4\pi^2}{3\varepsilon^2}, \\ 4\pi^2 \|\vartheta_t(\cdot, t)\|_2^2 &= \sum_{n \in \mathbb{Z}} \dot{H}_n^2(t) \leq 1 + 2\bar{n} + \sum_{|n| > \bar{n}} \left[ \frac{1}{\varepsilon \omega_n} + 2e^{t(\frac{2}{\varepsilon} - \varepsilon n^2)} \right]^2 \leq 1 + 2\bar{n} + \sum_{|n| > \bar{n}} \left[ \frac{2}{\varepsilon^2 \omega_n^2} + 8e^{2t(\frac{2}{\varepsilon} - \varepsilon n^2)} \right] \\ &\leq 3 + \frac{4}{\varepsilon} + \frac{4}{3\varepsilon^2} \sum_{n=\bar{n}+1}^{\infty} \frac{1}{n^2} + 8e^{\frac{4}{\varepsilon}t} \sum_{n \in \mathbb{Z}} e^{-2\varepsilon n^2 t} < 3 + \frac{4}{\varepsilon} + \frac{2\pi^2}{9\varepsilon^2} + 8e^{\frac{4t}{\varepsilon}} \theta\left(0, i\frac{2}{\pi}\varepsilon t\right), \\ 4\pi^2 \|\vartheta_{tx}(\cdot, t)\|_2^2 &= \sum_{n \in \mathbb{Z}} |\dot{H}_n(t)n|^2 \leq \bar{n}^4(\bar{n}^4 + 1) + \sum_{|n| > \bar{n}^4} n^2 \left[ \frac{1}{\omega_n \varepsilon} + \frac{h_n}{\omega_n} e^{t(\frac{2}{\varepsilon} - \varepsilon n^2)} \right]^2 \\ &\leq \bar{n}^4(\bar{n}^4 + 1) + \sum_{|n| > \bar{n}^4} \left[ \frac{2n^2}{\omega_n^2 \varepsilon^2} + 8n^2 e^{2t(\frac{2}{\varepsilon} - \varepsilon n^2)} \right] \leq \bar{n}^4(\bar{n}^4 + 1) + \frac{2}{\varepsilon^2} \sum_{|n| > \bar{n}^4} |n-1|^{-\frac{3}{2}} \\ &\quad + 16e^{\frac{4}{\varepsilon}t} \sum_{n \in \mathbb{Z}} n^2 e^{-2\varepsilon n^2 t} \leq \bar{n}^4(\bar{n}^4 + 1) + \frac{4}{\varepsilon^2} \zeta\left(\frac{3}{2}\right) - \frac{8}{\varepsilon} e^{\frac{4}{\varepsilon}t} \partial_t \left[ \theta\left(0, i\frac{2\varepsilon}{\pi}t\right) \right]. \quad \square \end{aligned}$$

In spite of (21), the Fourier series of  $\vartheta_x(\cdot, t)$  does not converge everywhere. Moreover that of  $\vartheta_t$  diverges for  $x = 2k\pi$  and  $t = 0$ . However the Fourier series obtained deriving termwise  $\vartheta$  (an arbitrary number of times) w.r.t.  $x, t$  define for all  $(x, t) \in \mathbb{R} \times I$  (time-dependent) periodic distributions (see e.g. [18]) w.r.t. the  $x$  variable, since their coefficients grow *slowly* with  $|n|$  (i.e. at most with a power law), by (22). The space  $S$  of test functions consists of

infinitely differentiable periodic functions. The coefficients  $g_n$  of the Fourier expansion

$$g(x) = \sum_{n \in \mathbb{Z}} g_n e^{inx}, \quad g_n := \frac{1}{2\pi} \int_0^{2\pi} dx g(x) e^{-inx}, \quad (43)$$

of  $g \in S$  have a *fast* decrease with  $|n|$  (i.e. faster than any power). By definition, applying  $\eta \in S'$  (the space of periodic distributions) to a  $g \in S$  gives

$$\langle \eta, g \rangle = \sum_{n \in \mathbb{Z}} \eta_{-n} g_n. \quad (44)$$

As said,  $H_n \rightarrow 0$ ,  $\dot{H}_n \rightarrow 1$  as  $t \rightarrow 0$ . Hence

$$\lim_{t \downarrow 0} \vartheta(x, t) = 0, \quad \lim_{t \downarrow 0} \vartheta_t(x, t) = \sum_{k \in \mathbb{Z}} \delta(x - 2\pi k) \quad (45)$$

in the sense of convergence in  $S'$ ; the rhs(45)<sub>2</sub> is the periodic delta function. In fact,  $\vartheta$  is the only  $t$ -dependent periodic distribution fulfilling (20), (45).

We can and shall use (44) as a definition of functional  $\eta$  also on less regular functions spaces. For instance, if  $g \in L^2([0, 2\pi])$  then (44) makes sense for  $\eta \in L^2([0, 2\pi])$ , and

$$\langle \eta, g \rangle = \int_0^{2\pi} \frac{dx}{2\pi} \eta(x) g(x). \quad (46)$$

The fundamental solution  $K(x, t)$  of the equation  $Lu = 0$  on  $\mathbb{R} \times \mathbb{R}^+$  was determined for  $a = 0$  in [2] and for  $a > 0$  in [19] in the form of quite complicated integrals involving the modified Bessel function of order zero.  $K(\cdot, t)$  is a Schwarz function for any  $t > 0$ . Since  $K(x, t) \rightarrow 0$ ,  $K_t(x, t) \rightarrow \delta(x)$  (in the sense of convergence of tempered distributions) as  $t \rightarrow 0$ , the present  $\vartheta$  must be related to  $K$  by

$$\vartheta(x, t) = \sum_{m \in \mathbb{Z}} K(x + 2m, t). \quad (47)$$

This is the analog of a property of Jacobi Theta function, which is the fundamental solution of the heat equation. Eq. (47) was used as a *definition* of  $\vartheta$  in the DBC case for  $a = 0$  in [5, 8]. However, here we prefer to work directly with the simpler and explicit definition (17), as done in [6] for the DBC case (alone).

### 3 Green functions and convolutions with them

We define the Green functions appropriate for the three boundary conditions:

$$\begin{aligned}
w^p(x, t; \xi) &:= \vartheta(x - \xi, t) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} H_n(t) e^{in(x - \xi)} && \text{PBC,} \\
w^d(x, t; \xi) &:= \vartheta(x - \xi, t) - \vartheta(x + \xi, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} H_n(t) \sin(nx) \sin(n\xi) && \text{DBC,} \\
w^n(x, t; \xi) &:= \vartheta(x - \xi, t) + \vartheta(x + \xi, t) = \frac{1 - e^{-at}}{\pi a} + \frac{2}{\pi} \sum_{n=1}^{\infty} H_n(t) \cos(nx) \cos(n\xi) && \text{NBC.}
\end{aligned} \tag{48}$$

For  $t > 0$  and  $w = w^p, w^d, w^n$  and the derivatives  $w_t, w_{tt}, w_{ttt}, \partial_x^2(\varepsilon w_t + w), \partial_x^2(\varepsilon w_{tt} + w_t), \partial_\xi^2(\varepsilon w_t + w)$  are continuous functions of  $x, t, \xi$ , of period  $2\pi$  w.r.t. both variables  $x, \xi$ , fulfilling

$$Lw = 0, \quad Lw_t = 0. \tag{49}$$

For all  $t$  they are  $t$ -dependent distributions w.r.t. both variables  $x, \xi$  fulfilling

$$w^d(k\pi, t; \xi) = w^d(x, t; k\pi) \equiv 0, \quad w^n(k\pi, t; \xi) = w^n(x, t; k\pi) \equiv 0 \tag{50}$$

for all  $k \in \mathbb{Z}$ , and analogous relations obtained deriving both sides w.r.t.  $t$ . Moreover,  $w = w^p, w^d, w^n$  fulfill limits analogous to (45).

Let  $C^{pk}$  be the space of (complex) functions of period  $2\pi$  continuous with their derivatives up to the  $k$ -th order. If  $g \in C^{p0}$  then  $g \in L^2([0, 2\pi])$  as well, what implies  $\|g\|_2^2 = \sum_{n \in \mathbb{Z}} |g_n|^2 < \infty$ ; whereas if  $g \in C^{p1}$  then, as known,  $\|g\|_1 = \sum_{n \in \mathbb{Z}} |g_n| < \infty$ , and the series (43) converges absolutely and uniformly to  $g(x)$  in all of  $\mathbb{R}$ . Similarly for  $g \in C^{pk}$  with  $k > 1$ . Let

$$C^{dk} := \{g \in C^{pk} \mid g(x) = g(\pi) = 0\}, \quad C^{mk} := \{g \in C^{pk} \mid g_x(x) = g_x(\pi) = 0\},$$

$$\begin{aligned}
w(x, t; \xi) &= w^p(x, t; \xi), & D &= [0, 2\pi], & \text{PBC,} \\
w(x, t; \xi) &= w^d(x, t; \xi), & D &= [0, \pi], & \text{DBC,} \\
w(x, t; \xi) &= w^n(x, t; \xi), & D &= [0, \pi], & \text{NBC.}
\end{aligned}$$

For  $w = w^p, w^d, w^n$  and resp.  $g \in C^{p0}, C^{d0}, C^{n1}$  let

$$w^g(x, t) := \langle w(x, t; \cdot), g \rangle = \int_D d\xi w(x, t; \xi) g(\xi). \tag{51}$$

$w^{pg}$  is just the convolution of  $\vartheta, g$ . By a straightforward calculation

$$\begin{aligned} w^{pg}(x, t) &= \sum_{n \in \mathbb{Z}} H_n(t) e^{inx} g_n, & g \in C^{p0}, \\ w^{dg}(x, t) &= \sum_{n=1}^{\infty} H_n(t) \sin(nx) g_n, & g \in C^{d0}, \\ w^{ng}(x, t) &= \frac{1-e^{-at}}{a} g_0 + \sum_{n=1}^{\infty} H_n(t) \cos(nx) g_n, & g \in C^{n1}. \end{aligned} \quad (52)$$

**Proposition 3.1** *For  $w^g = w^{pg}, w^{dg}, w^{ng}$  (with  $g \in C^{p0}, C^{d0}, C^{n1}$  respectively) the series (52) define for all  $t \geq 0$  a continuous function;  $w^g$  is real-valued if  $g$  is. For  $t > 0$  the derivatives  $w^g, w_t^g, w_{tt}^g, w_{ttt}^g, \partial_x^2(\varepsilon w_t^g + w^g), \partial_x^2(\varepsilon w_{tt}^g + w_t^g)$  are well-defined, are uniformly in  $x$  the sum of the corresponding term-by-term derived series and fulfill*

$$Lw^g = 0, \quad Lw_t^g = 0. \quad (53)$$

$w^g, w_t^g$  fulfill the ‘initial’ conditions

$$\lim_{t \downarrow 0} w^g(x, t) \equiv 0 \quad \text{uniformly in } x, \quad (54)$$

$$\lim_{t \downarrow 0} w_t^g(x, t) = g(x) \quad \text{uniformly in } x \quad \text{if } g \in C^1 \quad (55)$$

and the respective boundary conditions, more precisely:

$$\begin{aligned} w^{pg}(\cdot, t) &\in C^{p1}, & w^{dg}(\cdot, t) &\in C^{d1}, & w^{ng}(\cdot, t) &\in C^{n2}, & t &\geq 0, \\ w_t^{pg}(\cdot, t) &\in C^{p1}, & w_t^{dg}(\cdot, t) &\in C^{d1}, & w_t^{ng}(\cdot, t) &\in C^{n2}, & t &> 0. \end{aligned} \quad (56)$$

If in addition  $g$  has a continuous second derivative, then

$$\int_{\mathbb{D}} d\xi g(\xi) [\partial_x^2(\varepsilon w_t + w)](x, t; \xi) = \int_{\mathbb{D}} d\xi g''(\xi) [\varepsilon w_t + w](x, t; \xi). \quad (57)$$

The results in Proposition 3.1 generalize results of [6].

The regularity of  $w^g$  improves with that of  $g$ ; in particular, if  $g$  is infinitely differentiable, so is  $w^g$ . If  $g \notin C^1$ , for (55) to hold at  $x$  it suffices that left and right derivatives of  $g$  both exist at  $x$ , by standard wisdom about the Fourier series.

*Proof.* The mentioned series converge uniformly in  $x$  for  $t > 0$  by (22), (21) and Schwarz inequality in  $l^2(\mathbb{Z})$ . Eq. (53) follows from (20). One can check (57) just by noting that

the Fourier expansions of both sides coincide and converge, see (52). To prove (56) first note that each term of the Fourier series (52) fulfills the corresponding boundary conditions, then that the series and their term-by-term derivatives converge uniformly. As examples we prove (56)<sub>1</sub>, (56)<sub>4</sub> :  $w_x^{pg}(x, t) = i \sum_{n \in \mathbb{Z}} H_n(t) g_n n e^{inx}$ ,  $w_{tx}^{pg}(x, t) = i \sum_{n \in \mathbb{Z}} \dot{H}_n(t) g_n n e^{inx}$ ; using Schwarz inequality in  $l^2(\mathbb{Z})$  and (21) we find

$$\begin{aligned}
|w_x^{pg}(x, t)| &\leq \sum_{n \in \mathbb{Z}} |H_n(t) g_n n| \leq \left[ \sum_{n \in \mathbb{Z}} |H_n(t) n|^2 \right]^{\frac{1}{2}} \left[ \sum_{n \in \mathbb{Z}} |g_n|^2 \right]^{\frac{1}{2}} \\
&= 2\pi \|\vartheta_x(\cdot, t)\|_2 \|g\|_2 < \sqrt{2 + \frac{4}{\varepsilon} + \frac{2\pi^2}{3\varepsilon}} \|g\|_2 < \infty \quad t \geq 0, \\
|w_{tx}^{pg}(x, t)| &\leq \sum_{n \in \mathbb{Z}} |\dot{H}_n(t) g_n n| \leq \left[ \sum_{n \in \mathbb{Z}} |\dot{H}_n(t) n|^2 \right]^{\frac{1}{2}} \left[ \sum_{n \in \mathbb{Z}} |g_n|^2 \right]^{\frac{1}{2}} \\
&= 2\pi \|\vartheta_{tx}(\cdot, t)\|_2 \|g\|_2 < \infty \quad t > 0.
\end{aligned}$$

Next, from (24) and (18) it follows

$$4\pi^2 \|\vartheta(\cdot, t)\|_2^2 \equiv \sum_{n \in \mathbb{Z}} |H_n(t)|^2 \leq t \sum_{n \in \mathbb{Z}} |H_n(t)| \leq t N(t) \quad (58)$$

for all  $t$ . Therefore not only the sequence  $\{H_n(t)\}_{n \in \mathbb{Z}}$  is in  $l^2(\mathbb{Z})$  for all  $t$ , but its norm goes to 0 as  $t \rightarrow 0$ . Using again Schwarz inequality in  $l^2(\mathbb{Z})$  we find as a consequence

$$|w^{pg}(x, t)| \leq \sum_{n \in \mathbb{Z}} |H_n(t) g_n| \leq \left[ \sum_{n \in \mathbb{Z}} |H_n(t)|^2 \right]^{\frac{1}{2}} \left[ \sum_{n \in \mathbb{Z}} |g_n|^2 \right]^{\frac{1}{2}} < \sqrt{t N(t)} \|g\|_2.$$

This shows (54) in the PBC case. On the other hand, for any  $m \in \mathbb{N}$  it is

$$\begin{aligned}
|g(x, t) - w_t^{pg}(x, t)| &\leq \sum_{n=-m}^m \left| 1 - \dot{H}_n(t) \right| |g_n| + \sum_{|n| > m} \left| 1 - \dot{H}_n(t) \right| |g_n| \\
&\leq t \left[ \sum_{n=-m}^m 2h_n |g_n| + \sum_{\substack{n \in \mathbb{Z}: \\ \omega_n \in i\mathbb{R}}} |\Im(\omega_n)| |g_n| \right] + 2 \sum_{|n| > m} |g_n| < t(2h_m + n_+) \|g\|_1 + 2 \sum_{|n| > m} |g_n|,
\end{aligned}$$

where we have used the inequality  $|\Im(\omega_n)| < |n| < n_+$  following from (31) and the fact that the sequence  $\{h_n\}_{n \in \mathbb{N}}$  is increasing. For any  $\eta > 0$  choose  $m$  so large that  $2\sum_{|n| > m} |g_n| < \eta/2$ . Setting  $\delta := \eta/2(2h_m + n_+)\|g\|_1$  we find

$$|g(x, t) - w_t^{pg}(x, t)| < \eta \quad \forall x \in \mathbb{R}, \quad t < \delta. \quad (59)$$

This shows (55) in the PBC case.

Applying to  $g^d \in C^{d0}$  and  $g^n \in C^{n1}$  respectively odd, even extensions defined as in (12-13) (ignoring the  $t$  dependence there) one obtains a  $g^p \in C^{p0}$  such that  $w^{dg^d} = w^{pg^p}$  in  $[0, \pi]$  and a  $g^p \in C^{p1}$  such that  $w^{ng^n} = w^{pg^p}$  in  $[0, \pi]$ , respectively. This shows that (56), (54-55) in the DBC, NBC cases follow from (56), (54-55) in the PBC case.  $\square$

## 4 Existence and uniqueness

**Proposition 4.1** *Problem (6-7) is equivalent to the integral equation*

$$\begin{aligned} u(x, t) = & \int_{\mathbf{D}} d\xi \left\{ \left[ u_1 - \varepsilon u_0'' + a u_0 \right](\xi) w(x, t; \xi) + u_0(\xi) w_t(x, t; \xi) \right\} \\ & + \int_0^t d\tau \int_{\mathbf{D}} d\xi f[\xi, \tau, U(\xi, \tau)] w(x, t - \tau; \xi). \end{aligned} \quad (60)$$

*Proof.* Let  $L' := \partial_\tau^2 - a\partial_\tau - \partial_\xi^2(1 - \varepsilon\partial_\tau)$ . Assuming that  $u(x, t)$  solves (6)<sub>1</sub> it is straightforward to prove the identity [5]

$$\partial_\xi(u\tilde{w}_\xi - u_\xi\tilde{w} + \varepsilon u_\xi\tilde{w}_\tau - \varepsilon u\tilde{w}_{\xi\tau}) + \partial_\tau(u_\tau\tilde{w} - \varepsilon u_{\xi\xi}\tilde{w} + a u\tilde{w} - u\tilde{w}_\tau) - f\tilde{w} + uL'\tilde{w} = 0, \quad (61)$$

for any smooth functions  $u(\xi, \tau)$ ,  $\tilde{w}(\xi, \tau)$ . Choosing  $\tilde{w}(\xi, \tau) = w(x, t - \tau; \xi)$ , with  $w = w^p, w^d, w^n$  resp. in the PBC, DBC, NBC cases, the term  $uL'\tilde{w}$  becomes identically zero. Integrating (61) in  $d\xi$  over the respective domains  $\mathbf{D}$  we obtain

$$\begin{aligned} 0 = & \int_{\mathbf{D}} d\xi [\partial_\xi(u\tilde{w}_\xi - u_\xi\tilde{w} + \varepsilon u_\xi\tilde{w}_\tau - \varepsilon u\tilde{w}_{\xi\tau}) + \partial_\tau(u_\tau\tilde{w} - \varepsilon u_{\xi\xi}\tilde{w} + a u\tilde{w} - u\tilde{w}_\tau) - f\tilde{w}] \\ = & [u\tilde{w}_\xi - u_\xi\tilde{w} + \varepsilon u_\xi\tilde{w}_\tau - \varepsilon u\tilde{w}_{\xi\tau}]_{\xi=0}^{\xi=b} + \int_{\mathbf{D}} d\xi [\partial_\tau(u_\tau\tilde{w} - \varepsilon u_{\xi\xi}\tilde{w} + a u\tilde{w} - u\tilde{w}_\tau) - f\tilde{w}], \end{aligned}$$

where  $b = 2\pi$  in the PBC case and  $b = \pi$  in the DBC, NBC cases. The expression in the square bracket vanishes in all three cases by the periodicity of  $w^p, w_\tau^p, w_\xi^p, w_{\xi\tau}^p$  w.r.t. to  $\xi$  or



the boundary conditions (7), (50). By further integrating in  $d\tau$  over  $]\eta, t-\eta[$  (where  $\eta > 0$ ) we find

$$\begin{aligned} \int_{\eta}^{t-\eta} d\tau \int_{\mathbb{D}} d\xi f \tilde{w} &= \int_{\mathbb{D}} d\xi [u_{\tau} \tilde{w} - \varepsilon u_{\xi\xi} \tilde{w} + au \tilde{w} - u \tilde{w}_{\tau}]_{\tau=\eta}^{\tau=t-\eta} \\ &= \int_{\mathbb{D}} d\xi \left\{ [u_{\tau} - \varepsilon u_{\xi\xi} + au](\xi, t-\eta) w(x, \eta; \xi) - u(\xi, t-\eta) w_{\tau}(x, \eta; \xi) \right. \\ &\quad \left. - [u_{\tau} - \varepsilon u_{\xi\xi} + au](\xi, \eta) w(x, t-\eta; \xi) - u(\xi, \eta) w_{\tau}(x, t-\eta; \xi) \right\}. \end{aligned}$$

By Schwarz inequality

$$\left| \int_{\mathbb{D}} d\xi [u_{\tau} - \varepsilon u_{\xi\xi} + au](\xi, t-\eta) w(x, \eta; \xi) \right| \leq \| [u_{\tau} - \varepsilon u_{\xi\xi} + au](\cdot, t-\eta) \|_2 \| \vartheta(\cdot, \eta) \|_2;$$

by (58), this goes to zero as  $\eta \rightarrow 0$ . Letting  $\eta \rightarrow 0$ , by (6), (55) we find that  $u$  satisfies the integral equation (60).

Conversely, assume  $u$  solves (60). The rhs of the first line of (60) is nothing but  $w^{u_1+au_0-\varepsilon u_0''} + w_t^{u_0}$ ; it fulfills the respective boundary conditions by (56). Also the second line fulfills the respective boundary conditions, by (50). Next, let us check that  $u$  fulfills  $Lu = f$ .  $L$  applied to the first line of (60) gives zero, by (53). Denoting as  $\mathcal{I}$  the second line, we find

$$\begin{aligned} \mathcal{I}_t &= \int_{\mathbb{D}} d\xi f[\xi, t, U(\xi, t)] w(x, 0; \xi) + \int_0^t d\tau \int_{\mathbb{D}} d\xi f[\xi, \tau, U(\xi, \tau)] w_t(x, t-\tau; \xi) \\ &= \int_0^t d\tau \int_{\mathbb{D}} d\xi f[\xi, \tau, U(\xi, \tau)] w_t(x, t-\tau; \xi) \end{aligned} \tag{62}$$

$$\begin{aligned} \mathcal{I}_{tt} &= \int_{\mathbb{D}} d\xi f[\xi, t, U(\xi, t)] w_t(x, 0; \xi) + \int_0^t d\tau \int_{\mathbb{D}} d\xi f[\xi, \tau, U(\xi, \tau)] w_{tt}(x, t-\tau; \xi) \\ &= f[x, t, U(x, t)] + \int_0^t d\tau \int_{\mathbb{D}} d\xi f[\xi, \tau, U(\xi, \tau)] w_{tt}(x, t-\tau; \xi) \Rightarrow \\ Lu &= L\mathcal{I} = f[x, t, U(x, t)] + \int_0^t d\tau \int_{\mathbb{D}} d\xi f[\xi, \tau, U(\xi, \tau)] (Lw)(x, t-\tau; \xi) \\ &= f[x, t, U(x, t)] \end{aligned} \tag{63}$$

as claimed. We have used (54) in the second equality, (55) in the fourth, (53) in the sixth. Finally, let us check that  $u$  fulfills the required initial conditions. Taking the limit  $t \downarrow 0$  and

using (54), (55) it is straightforward to show that (60) implies  $u(x, 0) = u_0(x)$ . We now evaluate  $u_t(x, t)$ :

$$\begin{aligned}
u_t(x, t) &= \int_{\mathbf{D}} d\xi \left\{ \left[ u_1 + au_0 - \varepsilon u_0'' \right](\xi) w_t(x, t; \xi) + u_0(\xi) w_{tt}(x, t; \xi) \right\} + \mathcal{I}_t \\
&= \int_{\mathbf{D}} d\xi \left\{ \left[ u_1 + au_0 - \varepsilon u_0'' \right](\xi) w_t(x, t; \xi) + u_0(\xi) [\partial_x^2(\varepsilon w_t + w) - a w_t](x, t; \xi) \right\} + \mathcal{I}_t \\
&= \int_{\mathbf{D}} d\xi \left\{ \left[ u_1 - \varepsilon u_0'' \right](\xi) w_t(x, t; \xi) + u_0''(\xi) [\varepsilon w_t + w](x, t; \xi) \right\} + \mathcal{I}_t \\
&= \int_{\mathbf{D}} d\xi [u_1(\xi) w_t(x, t; \xi) + u_0''(\xi) w(x, t; \xi)] + \int_0^t \int_{\mathbf{D}} d\xi f[\xi, \tau, U(\xi, \tau)] w_t(x, t - \tau; \xi);
\end{aligned}$$

we have used (62) in the first equality, (49) in the second, (57) in the third. Taking the limit  $t \downarrow 0$  and using (54), (55) we find  $u_t(x, 0) = u_1(x)$ , as claimed.  $\square$

If  $f = f(x, t)$ , the rhs(60) gives the unique explicit solution of (6-7). Otherwise, to deal with the integro-differential equation (60) it is convenient to reformulate it in any finite time interval  $[0, T]$  as the fixed point equation

$$\mathcal{T}u = u. \quad (64)$$

$\mathcal{T}$  is the linear map  $\mathcal{T} : \mathcal{B} \mapsto \mathcal{B}$  defined by

$$\begin{aligned}
\mathcal{B} &:= \{v(x, t) \text{ of period } 2\pi \mid v, v_x, v_t \in C(D_T)\}, \quad D_T := \mathbf{D} \times [0, T] \\
[\mathcal{T}v](x, t) &:= \int_{\mathbf{D}} d\xi \left\{ \left[ u_1 + au_0 - \varepsilon u_0'' \right](\xi) w(x, t; \xi) + u_0(\xi) w_t(x, t; \xi) \right\} \\
&\quad + \int_0^t \int_{\mathbf{D}} d\xi f[\xi, \tau, V(\xi, \tau)] w(x, t - \tau; \xi).
\end{aligned} \quad (65)$$

$\mathcal{B}$  is a Banach space w.r.t. the norm

$$\|v\|_{\lambda, T} := \max_{D_T} |e^{-\lambda t} v(x, t)| + \max_{D_T} |e^{-\lambda t} v_x(x, t)| + \max_{D_T} |e^{-\lambda t} v_t(x, t)|, \quad (66)$$

where  $\lambda$  is some positive constant we fix below. We shall assume (in all three cases) that  $f$  is continuous in  $(x, t, v) \in D \times I \times \mathbb{R}^3$  and satisfies a Lipschitz condition w.r.t.  $v_1, v_2, v_3$ :

$$|f(x, t, v) - f(x, t, y)| \leq \mu(|v_1 - y_1| + |v_2 - y_2| + |v_3 - y_3|), \quad \mu \in \mathbb{R}^+. \quad (67)$$

Note that (67) remains true for  $\hat{f}$  after the transformations  $f \mapsto \hat{f}$  defined in (8) and  $f \mapsto \tilde{f}$  defined in (10). We can now state the main result of the present paper.

**Theorem 4.1** *If  $f = f(x, t, v)$  is continuous and Lipschitz with respect to  $v_1, v_2, v_3$ , then the nonlinear problem (1) with Dirichlet, Neumann, or pseudoperiodic boundary conditions (2) has a unique solution in all  $D \times [0, \infty[$ .*

*Proof.* If  $a < 0$  we apply the change of variables (10) and reduce the existence and uniqueness theorem for the case  $a < 0$  to the one for the case  $a = 0$ . So it suffices to prove the theorem for  $a \geq 0$ . Let  $\Delta f(\xi, \tau) := f[\xi, \tau, V_1(\xi, \tau)] - f[\xi, \tau, V_2(\xi, \tau)]$ . Using (67) we find for  $(\xi, \tau) \in D_T$

$$|\Delta f(\xi, \tau)| e^{-\lambda \tau} \leq \mu \|v_1 - v_2\|_{\lambda, T}, \quad \|\Delta f(\cdot, \tau)\|_2^2 = \int_D \frac{d\xi}{2\pi} \Delta f^2(\xi, \tau) \leq \mu^2 \|v_1 - v_2\|_{\lambda, T}^2 e^{2\lambda \tau}. \quad (68)$$

From (60) and (54) we obtain for  $(x, t) \in D_T$

$$\begin{aligned} [\mathcal{T}v_1 - \mathcal{T}v_2](x, t) &= \int_0^t d\tau \int_D d\xi w(x, t-\tau; \xi) \Delta f(\xi, \tau), \\ [\mathcal{T}v_1 - \mathcal{T}v_2]_x(x, t) &= \int_0^t d\tau \int_D d\xi w_x(x, t-\tau; \xi) \Delta f(\xi, \tau), \\ [\mathcal{T}v_1 - \mathcal{T}v_2]_t(x, t) &= \int_D d\xi w(x, 0; \xi) \Delta f(\xi, t) + \int_0^t d\tau \int_D d\xi w_t(x, t-\tau; \xi) \Delta f(\xi, \tau) \\ &= \int_0^t d\tau \int_D d\xi w_t(x, t-\tau; \xi) \Delta f(\xi, \tau) \end{aligned} \quad (69)$$

Inequality (18) implies for all  $(x, t) \in D_T$   $\int_D |w(x, t; \xi)| d\xi \leq 2N(t)$  and, by (68-69),

$$\begin{aligned} |[\mathcal{T}v_1 - \mathcal{T}v_2](x, t)| e^{-\lambda t} &\leq \int_0^t d\tau e^{-\lambda t} \int_D d\xi |w(x, t-\tau; \xi)| |\Delta f(\xi, \tau)| \\ &\leq \mu \|v_1 - v_2\|_{\lambda, T} \int_0^t e^{-\lambda(t-\tau)} d\tau \int_D |w(x, t-\tau; \xi)| d\xi \\ &\leq \mu \|v_1 - v_2\|_{\lambda, T} \int_0^t e^{-\lambda(t-\tau)} 2N(t-\tau) d\tau \\ &\leq \frac{2\mu M'}{\lambda} \|v_1 - v_2\|_{\lambda, T}, \quad M' := M + \begin{cases} 0 & \text{DBC,} \\ a^{-1} & \text{PBC, NBC and } a \neq 0, \\ \lambda^{-1} & \text{PBC, NBC and } a = 0. \end{cases} \end{aligned} \quad (70)$$

On the other hand,  $\theta(0, i\eta)$  is a positive strictly decreasing function of  $\eta > 0$  fulfilling the property  $\theta(0, \eta) (-i\eta)^{\frac{1}{2}} = \theta\left(\frac{z}{\eta}, \frac{-1}{\eta}\right)$ , (see e.g. [17], p. 33). For  $\tau \in [0, T]$  it follows

$$\left(\frac{2\varepsilon\tau}{\pi}\right)^{\frac{1}{2}} \theta\left(0, i\frac{2}{\pi}\varepsilon\tau\right) = \theta\left(0, \frac{i\pi}{2\varepsilon\tau}\right) \leq \theta\left(0, \frac{i\pi}{2\varepsilon T}\right) =: \frac{1}{4} \left(\frac{\varepsilon}{2\pi}\right)^{\frac{1}{2}} \Theta^2;$$

hence and from (21)<sub>2</sub> we find for  $\tau \in [0, T]$

$$2\pi \|\vartheta_\tau(\cdot, \tau)\|_2 \leq \sqrt{\kappa + 8e^{\frac{4\tau}{\varepsilon}} \theta\left(0, i\frac{2}{\pi}\varepsilon\tau\right)} < \sqrt{\kappa} + e^{\frac{2\tau}{\varepsilon}} \sqrt{8\theta\left(0, i\frac{2}{\pi}\varepsilon\tau\right)} \leq \sqrt{\kappa} + \Theta e^{\frac{2\tau}{\varepsilon}} \tau^{-\frac{1}{4}}. \quad (71)$$

Using Schwarz inequality and (21), (48), (68), (71) eq. (69) imply

$$\begin{aligned} |[\mathcal{T}v_1 - \mathcal{T}v_2]_x(x, t)| e^{-\lambda t} &\leq \int_0^t d\tau \left| \int_D d\xi w_x(x, t-\tau; \xi) \Delta f(\xi, \tau) \right| e^{-\lambda t} \\ &\leq \int_0^t d\tau 2\pi \|\vartheta_x(\cdot, t-\tau)\|_2 \|\Delta f(\cdot, \tau)\|_2 e^{-\lambda t} \leq \mu \|v_1 - v_2\|_{\lambda, T} \left(2 + \frac{12+2\pi^2}{3\varepsilon}\right)^{\frac{1}{2}} \int_0^t d\tau e^{-\lambda(t-\tau)} \\ &\leq \frac{\mu}{\lambda} \left(2 + \frac{12+2\pi^2}{3\varepsilon}\right)^{\frac{1}{2}} \|v_1 - v_2\|_{\lambda, T} \end{aligned} \quad (72)$$

$$\begin{aligned} |[\mathcal{T}v_1 - \mathcal{T}v_2]_t(x, t)| e^{-\lambda t} &\leq \int_0^t d\tau \left| \int_D d\xi w_t(x, t-\tau; \xi) \Delta f(\xi, \tau) \right| e^{-\lambda t} \\ &\leq \int_0^t d\tau 2\pi \|\vartheta_t(\cdot, t-\tau)\|_2 \|\Delta f(\cdot, \tau)\|_2 e^{-\lambda t} \leq \mu \|v_1 - v_2\|_{\lambda, T} \int_0^t d\tau \left(\sqrt{\kappa} + \Theta e^{\frac{2\tau}{\varepsilon}} \tau^{-\frac{1}{4}}\right) e^{-\lambda\tau} \\ &\leq \mu \left[\frac{\sqrt{\kappa}}{\lambda} + \left(\lambda - \frac{2}{\varepsilon}\right)^{-\frac{3}{4}} \Theta \Gamma\left(\frac{3}{4}\right)\right] \|v_1 - v_2\|_{\lambda, T} \end{aligned} \quad (73)$$

[in the last step we have assumed  $\varepsilon\lambda > 2$  and used the gamma function  $\Gamma(z) = \int_0^\infty dy e^{-y} y^{z-1}$ ]. Eq. (70-73) imply

$$\|\mathcal{T}v_1 - \mathcal{T}v_2\|_{\lambda, T} \leq \left\{ \frac{\mu}{\lambda} \left[ 2M' + \left(2 + \frac{12+2\pi^2}{3\varepsilon}\right)^{\frac{1}{2}} + \sqrt{\kappa} \right] + \left(\lambda - \frac{2}{\varepsilon}\right)^{-\frac{3}{4}} \Theta \Gamma\left(\frac{3}{4}\right) \right\} \|v_1 - v_2\|_{\lambda, T}. \quad (74)$$

Hence  $\mathcal{T}$  is a contraction of  $\mathcal{B}$  into itself provided we choose  $\lambda > 2/\varepsilon$  so large that the coefficient of  $\|v_1 - v_2\|_{\lambda, T}$  at the rhs(74) is smaller than 1. Then, applying the fixed point theorem we find that there exists a unique solution of the problem  $\mathcal{T}u = u$  in  $\mathcal{B}$ , i.e. of (60) in the time interval  $[0, T]$ , for any  $T > 0$ , and therefore in all  $I = [0, \infty[$ .  $\square$

The existence and regularity of the solution *for all*  $t > 0$  crucially depends on the assumption that  $f$  fulfills the Lipschitz condition (67). As known, if we assumed  $f$  to fulfill Lipschitz condition (67) only *locally*<sup>5</sup>, then in general the fixed point theorem would be applicable only for a not too large  $T$ ; as a consequence, one could not exclude the occurrence of *blow-up's* [20], i.e. singularities of  $u$  or its derivatives, for sufficiently large  $t$ .

---

<sup>5</sup>If for any bounded set  $\Omega \subset D \times I \times \mathbb{R}$  there exists a constant  $\mu$  *depending on*  $\Omega$  such that for any  $(x, t, u_1), (x, t, u_2) \in \Omega$  (67) is satisfied, then  $f$  is said to satisfy a *local* Lipschitz condition w.r.t.  $u$ . Similarly if  $f$  depends on  $x, t, u, u_x, u_{xx}, u_t$ .

## References

- [1] P. Renno, *On some viscoelastic models*, Atti Acc. Lincei Rend. Fis. **75**, 1-10 (1983).
- [2] B. D'Acunto, P. Renno, *On Some Nonlinear Visco-elastic Models*, Ricerche di Matematica, **41**, 101-122 (1992).
- [3] M. De Angelis, P. Renno, *Existence, uniqueness and a priori estimates for a non linear integro - differential equation* , Ricerche di Matematica, **57**, 95-109 (2008).
- [4] M. De Angelis, A. Maio, E. Mazziotti, *Existence and uniqueness results for a class of non linear models*, Mathematical Physics models and Engineering sciences **8**, pp. 191-202. Eds: P. Renno. Liguori, Napoli (2008).
- [5] B. D'Acunto, A. D'Anna, *Stability for a third order Sine-Gordon equation*, Rend. Mat. Serie **VII**, Vol. 18, 347-365 (1998).
- [6] M. De Angelis, *Asymptotic analysis for the strip problem related to a parabolic third-order operator*, Appl. Math. Lett. **14**, 425-430 (2001).
- [7] A. D'Anna, G. Fiore *Stability and attractivity for a class of dissipative phenomena*, Rend. Mat. Serie **VII**, Vol. 21, 191-206 (2001).
- [8] A. D'Anna, G. Fiore, *Global Stability properties for a class of dissipative phenomena via one or several Liapunov functionals*, Nonlinear Dyn. Syst. Theory **5**, 9-38 (2005).
- [9] Josephson B. D. *Possible new effects in superconductive tunneling*, Phys. Lett. **1** (1962), 251-253; *The discovery of tunneling supercurrents*, Rev. Mod. Phys. B **46** (1974), 251-254; and references therein.
- [10] A. Barone, G. Paternó *Physics and Applications of the Josephson Effect*, Wiley-Interscience, New-York (1982); and references therein.
- [11] P. I. Christiansen, A. C. Scott, M. P. Sorensen, *Nonlinear Science at the Dawn of the 21st Century*, Lecture Notes in Physics **542**, Springer, Berlin (2000).
- [12] J. A. Morrison, *Wave propagations in rods of Voigt material and visco-elastic materials with three-parameters models*, Quart. Appl. Math., **14**, 153-169 (1956).
- [13] A. Morro, L. E. Payne, B. Straughan *Decay, growth, continuous dependence and uniqueness results of generalized heat theories*, Appl. Anal. **38**, 231-243 (1990).

- [14] N. Flavin, S. Rionero, *Qualitative Estimates for Partial Differential Equations*, CRC Press, Boca Raton - Florida (1996).
- [15] H. Lamb, *Hydrodynamics*, Cambridge University Press, Cambridge, 1959.
- [16] R. Nardini, *Soluzione di un problema al contorno della magneto-idrodinamica* (Italian), Ann. Mat. Pura Appl. **35** (1953), 269-290.
- [17] David Mumford, *Tata Lectures on Theta I* (1983), Birkhäuser, Boston, 1983.
- [18] R. Beals, *Advanced mathematical analysis: periodic functions and distributions, complex analysis, Laplace transform and applications*, 230 pp. Springer-Verlag, Berlin (1973).
- [19] B. D'Acunto, M. De Angelis, P. Renno, *Fundamental solutions for a dissipative operator*, Rend. Acc. Sc. fis. mat. Napoli **LXIV**, 295-314 (1997).
- [20] S. Alinhac, *Blowup for nonlinear hyperbolic equations*, 112 pp. Progress in nonlinear differential equations and their applications **17**, Birkhauser, Boston (1995).